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Impact of Sequential Disorder on Scaling Behavior of Boarding Time

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Airplane boarding process is an example where disorder properties of the system are relevant to emergence of universality classes. Based on a simple model, we present a systematic analysis of finite-size effects in boarding time, and propose a comprehensive view of the role of sequential disorder in the scaling behavior of boarding time against the plane size. Using numerical simulations and mathematical arguments, we find how the scaling behavior depends on the number of seat columns and the range of sequential disorder. Our results show that new scaling exponents can arise as disorder is localized to varying extents.

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How disorder properties affect the scaling behavior of a characteristic time scale is a problem studied in various areas of physics. For example, dynamic exponents of surface growth models [1] are known to depend on the geometry of substrate and the type of disorder. In complex networks, consensus time of opinion dynamics [2] and first-passage time of random walk [3] scale differently with the system size depending on the level of structural heterogeneity. Those studies reveal the subtle interplay between dynamics and disorder that gives rise to different universality classes of scaling behaviors, each of which is succinctly characterized by its scaling exponents. Expanding the list of those universality classes and clarifying their origins has been established as a general framework frequenty employed by physicists.

Airplane boarding process provides another interesting example where this framework can be applied. The average time required for all passengers to get seated, or *average boarding time* $\langle T \rangle$, may scale differently with the plane size N depending on sequential disorder of passengers, which is controlled by the airline's boarding policy and each individual's gate arrival time. However, most studies of boarding process were limited to the practical problem of reducing boarding time at a fixed plane size [4–10], which provides only fragmented knowledge about performances of different boarding policies in prescribed situations. To understand the general nature of boarding process, we should pay attention to its scaling properties, which came to be studied only recently with inconsistent conclusions. While analytical studies by Bachmat *et al.* [11–14] supported $\langle T \rangle \sim N^{1/2}$, a numerical study by Frette and Hemmer [15] based on a simple model (see Fig. 1) reported $\langle T \rangle \sim N^{0.69}$. This was again contradicted by Bernstein [16], who observed that the same model yields $\langle T \rangle \sim N^{1/2}$ after filtering out finite-size effect.

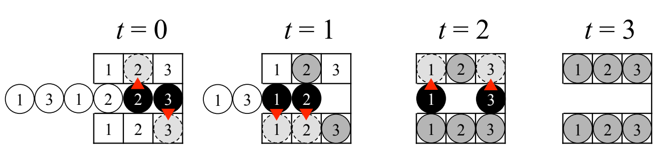

FIG. 1: Snapshots of boarding in an airplane with $N = 3$ rows and $c = 2$ columns of seats, where each passenger (circle) is labeled by the row index of its seat. At each time step, passengers are waiting in line (white), sitting down (black), or staying seated (gray).

In this Letter, we resolve the conflict between different exponents by showing that $T \sim N^{1/2}$ is the correct description of the previous results if we carefully interpret the relevant mathematical theorems [17, 18]. We also propose that different scaling behaviors are possible if the range of disorder in the passenger sequence is localized to varying extents, which gives a more comprehensive picture of the problem. To support these claims, we generalize the boarding model proposed by [15], incorporating additional parameters for the structure of the airplane and disorder in the passenger sequence.

Model — We consider an airplane with N rows and c columns of seats along a one-dimensional aisle. Each seat is labeled with an index r if it is in the r-th row from the front. The aisle is discretized into N sites, each of which cannot hold more than one passenger at once. At $t = 0$, cN passengers enter the plane from the front in a sequence $\{r_1, r_2, \ldots, r_{cN}\}$, where r_i denotes the seat index of the i-th passenger to enter. Passengers' positions are synchronously updated. Every passenger moves along the aisle front-to-back, instantly crossing successive empty sites until blocked by an occupied site or reaching the row of its assigned seat. If the latter is the case, sitting down takes one time step for every passenger. Continuing the process, all passengers would get seated at $t = T$, which we call boarding time (see Fig. 1).

Since the boarding process is deterministic in the

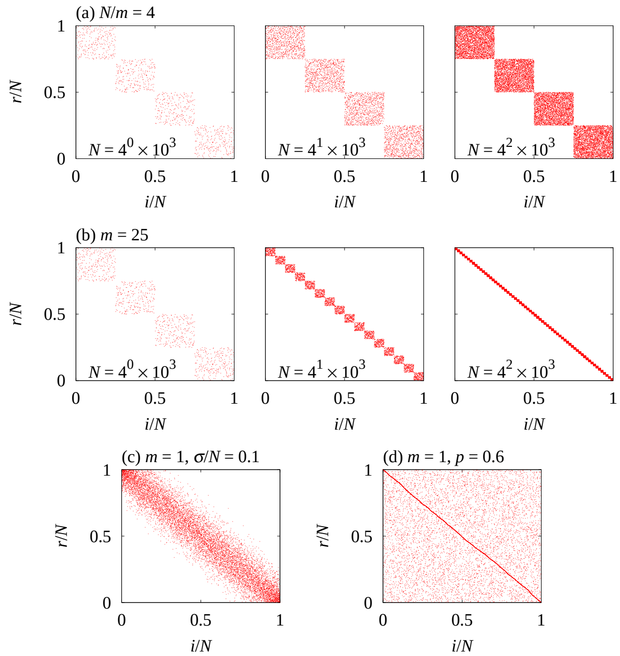

FIG. 2: Examples of the passenger sequence $\{r_i\}$. As $N \to \infty$, (a) fixed N/m means globalized disorder; (b) fixed m means localized disorder; (c) Gaussian shuffling gives either purely globalized or purely localized disorder; (d) uniform shuffling allows coexistence of both kinds of disorder. We set $N = 10^4$ in (c) and (d).

model, T is determined by the initial passenger sequence. Lack of strict control over the sequence implies that it is more meaningful to deal with $\langle T \rangle$ for an ensemble of sequences rather than T for a particular sequence [19]. As one can understand from Fig. 2, such an ensemble can be represented as a probability density function (PDF) $p(i/N, r/N)$, where i/N and r/N denote sequential and seat indices of passengers normalized by N, respectively [11–14]. This representation enables us to utilize the following mathematical theorem [17, 18] that has profound implications for the scaling behavior of $\langle T \rangle$.

Theorem — If (x_α, y_α), $\alpha = 1, \dots, N$ are pairs of real numbers with $0 \le x_\alpha \le 1$ and $0 \le y_\alpha \le 1$, we say that a subsequence $\{(x_{i_1}, y_{i_1}), \dots, (x_{i_l}, y_{i_l})\}$ is an *increasing subsequence* if

$$x_{i_j} < x_{i_{j+1}} \text{ and } y_{i_j} < y_{i_{j+1}} \text{ for } j = 1, \dots, l-1$$

where i_j is a sequence of non-repeated indices between 1 and N. If the pairs (x_α, y_α) are generated from a finite PDF $p(x, y)$, the length of the longest increasing subsequence asymptotically scales as $N^{1/2}$.

The increasing subsequence in the theorem can be translated as the blocking subsequence of passengers, in which one passenger *blocks* the next one if the latter cannot reach its seat unless the former is seated [11, 14]. Since the length of the longest blocking subsequence is equal to boarding time T, the theorem indicates that T (and $\langle T \rangle$ as well) scales as $N^{1/2}$ if the PDF remains finite in the asymptotic limit $N \to \infty$. The finitude of the PDF is equivalent to the broad range of disorder in the passenger sequence, which is comparable with the plane size and might be described as "globalized". In other words, $\langle T \rangle \sim N^{1/2}$ is guaranteed if sequential disorder is globalized. Meanwhile, different scaling behaviors might arise if some part of the PDF diverges due to the presence of "localized" disorder in the passenger sequence.

The range of sequential disorder is primarily determined by boarding policies. A boarding policy is a set of constraints imposed on the initial passenger sequence. Here we focus on back-to-front policies with equal-sized boarding groups. According to such policies, passengers are divided into boarding groups of m rows each (m is a divisor of N), so that the n-th group to enter the plane consists of passengers whose seat indices satisfy $N - nm + 1 \le r_i \le N - (n-1)m$. Consequently, the passenger sequence is sorted back-to-front on a scale larger than groups, but remains randomized within each group. The range of sequential disorder is given by m. If N/m (m) is fixed as $N \to \infty$, sequential disorder is purely globalized (localized), as illustrated in Fig. 2(a) and (b), respectively.

Meanwhile, some passengers would naturally deviate from the boarding policy due to their impunctuality. This justifies the incorporation of arrival time fluctuations as another determinant of sequential disorder. We consider Gaussian shuffling as a model of such fluctuations, which is defined as follows: we add an i.i.d. Gaussian random number η_i of zero mean and of variance σ^2 to the sequential index i of each passenger, and then sort the sequence in the increasing order of $i + \eta_i$ (see Fig. 2(c)). As long as m is finite, Gaussian shuffling makes sequential disorder purely localized (globalized) if σ (σ/N) is fixed in the asymptotic limit. Alternatively, we can also consider uniform shuffling, in which every passenger is randomly relocated in the sequence with probability p (see Fig. 2(d)). If m is finite, uniform shuffling allows localized and globalized disorder to coexist until the sequence is completely randomized at $p = 1$.

Now that we have a collection of parameters controlling the range of sequential disorder, we are ready to test numerically whether $\langle T \rangle \sim N^{1/2}$ is true for purely globalized disorder, and to examine unkown scaling behaviors in the presence of localized disorder. We use the method of effective scaling exponent [20] to clarify the asymptotic scaling, which is effectively the same as the method adopted by [16].

Single-column airplane — We first consider a fully reserved plane with only a single column ($c = 1$) of seats. Although a single-column plane hardly exists in reality, it is not merely a theoretical construct. For example, consider a boarding policy that fills the window seats first and the aisle seats later. The policy separates the passengers to be seated in the same row very far from

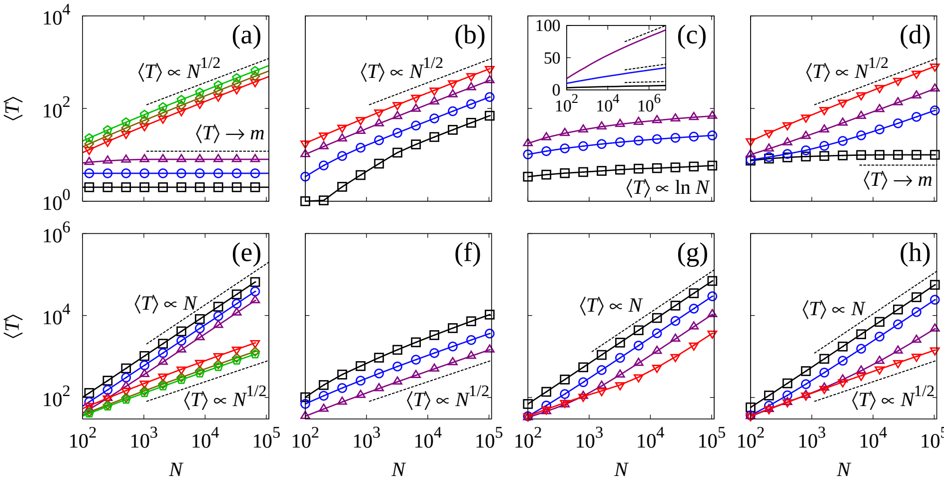

FIG. 3: (Color online) Asymptotic scaling behaviors of $\langle T\rangle$ in a fully reserved plane. When $c = 1$, from bottom to top, the curves represent (a) $m = 2, 4, 8, N/4, N/2, N$ without shuffling, and $m = 10$ with (b) $\sigma/N = 0.001, 0.01, 0.1, 1$, (c) $\sigma = 1, 10, 100$ (inset: semi-log plots of the main), (d) $p = 0, 0.01, 0.1, 1$. When $c = 2$, from top to bottom, (e) $m = 1, 4, 16, N/16, N/4, N$ without shuffling, and $m = 5$ with (f) $\sigma/N = 0.001, 0.01, 0.1$, (g) $\sigma = 1, 10, 100, 1000$, (h) $p = 0, 0.5, 0.9, 1$. Each data point is averaged over at least 100 samples.

each other, so there is effectively no interference between them. In such a case, even a multi-column plane can be regarded as a single-column plane.

The scaling behaviors of $\langle T\rangle$ for $c = 1$ are shown in Fig. 3(a)–(d). Without arrival time fluctuations, $\langle T\rangle$ scales as $N^{1/2}$ if N/m is fixed, while it saturates to m if m is fixed (see Fig. 3(a)). Gaussian shuffling at finite m produces similar but slightly different scalings: the $N^{1/2}$ scaling for fixed σ/N (see Fig. 3(b)) and the $\ln N$ scaling for fixed σ (see Fig. 3(c)). Since both saturation and log scaling are slower than any algebraic scaling, they can be collectively labeled as N^0 scalings. Thus, our results confirm $\langle T\rangle \sim N^{1/2}$ for purely globalized disorder, while finding $\langle T\rangle \sim N^0$ for purely localized disorder. When those two kinds of disorder coexist, the effect of localized disorder predominates, as implied by the $N^{1/2}$ scaling observed whenever $p > 0$ (see Fig. 3(d)).

We provide an argument for the origin of this new scaling in the special case when there are no arrival time fluctuations. As $N \to \infty$, the passenger sequence almost certainly contains a boarding group whose configuration is exactly front-to-back, *e.g.*, $\{1, 2, \ldots, m-1, m\}$. Since boarding time for this particular group cannot be less than m, so is boarding time of the entire passenger sequence, *i.e.*, $\lim_{N\to\infty} T \geq m$. With a lower bound of T thus obtained, we also build an argument for its upper bound. Let us choose one passenger from the sequence. When the boarding begins, the passenger has at most $m + n_< - 1$ sites to go before reaching its seat, where $n_<$ denotes the number of passengers in the same group whose seat indices are smaller. Since those $n_<$ passengers are closer to their seats, they would get seated before the chosen passenger, leaving empty sites in the aisle. The chosen passenger advances along the aisle by at least one site per unit time, and can move further if empty sites appear on its way. Hence, the chosen passenger can reach the reserved seat within $m - 1$ time steps and spends one more time step to sit down. Since the choice of the passenger was arbitrary, individual boarding time for any passenger is not greater than m, and so is the total boarding time, *i.e.*, $T \leq m$. Combining both upper and lower bounds, we obtain $\lim_{N\to\infty} T = m$.

Before moving on to the multi-column case, we remark that dividing into smaller groups always reduces boarding time if $c = 1$, as previously reported by [16]. This is the case when keeping the passenger sequence ordered is always good for saving time. However, we emphasize that the benefits of group division, in terms of scaling, are not very robust against disorder caused by arrival time fluctuations. A slightest hint of globalized disorder can revert the scaling behavior to that of the random boarding policy. Hence, division into small boarding groups is not very beneficial, if one aims to reduce the scaling exponent of boarding time.

Multi-column airplane — As a next step, we consider a fully reserved plane with multiple columns ($c > 1$) of seats, which is a very typical situation encountered in reality. Without loss of generality, we only focus on the boarding process of a two-column ($c = 2$) plane since the scaling behaviors does not change in the other cases.

Figure 3(e)–(h) show the scaling behaviors of $\langle T\rangle$ at $c = 2$. If no shuffling is applied, $\langle T\rangle$ scales as $N^{1/2}$ (N) if N/m (m) is fixed (see Fig. 3(e)). Gaussian shuffling at finite m results in exactly the same scalings (see Fig. 3(f) and (g)), implying $\langle T\rangle \sim N^{1/2}$ ($\langle T\rangle \sim N$) for purely globalized (localized) disorder. If both kinds of disorder are mixed together, the effect of globalized disorder predominates, as the linear scaling for $p < 1$ indicates (see Fig. 3(h)).

For the special case when no shuffling is involved, the origin of the linear scaling can be explained as follows. Since the probability of a boarding group having the configuration like $\{1, 2, \ldots, m, 1, 2, \ldots, m\}$ is finite, the number of such boarding groups grows linearly with N. When the first half of such a group begin to take seats, its later half completely occupy the rows reserved for the next group. This means that the next group cannot reach their seats for at least one time step, which implies that T

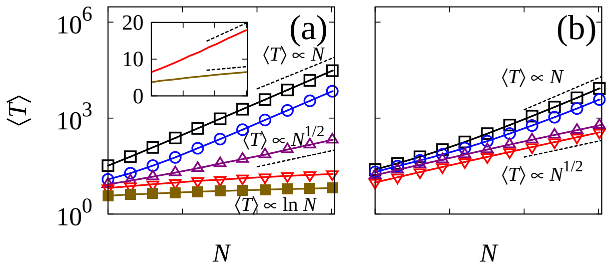

FIG. 4: (Color online) Asymptotic scaling behaviors of $\langle T \rangle$ in a partially reserved two-column ($c = 2$) airplane with group size $m = 5$. For the curves from top to bottom, (a) $\phi = 0.25, 0.45, 0.5, 0.55, 0.75$ without shuffling (inset: semi-log plots of the lowest two curves in the main) and (b) $\phi = 0.25, 0.35, 0.5, 0.55$ with uniform shuffling $p = 0.5$. Each data point is averaged over at least 100 samples.

TABLE I: Summary of asymptotic scalings of $\langle T \rangle$. Note that N^0 stands for both saturation to a finite value and logarithmic scaling. The gray arrow indicates that vacancy effect changes the multi-column scaling to the single-column one at ϕ_c.

Type of disorder	Number of columns Single ($c = 1$)	Multiple ($c > 1$)
Localized ($p = 0, \sigma \sim N^0$)	N^0	N^1
Mixed ($0 < p < 1$)	$N^{1/2}$	N^1
Globalized ($p = 1, \sigma \sim N^1$)	$N^{1/2}$	$N^{1/2}$

Vacancy ϕ: Transition at $\phi_c = 1 - 1/c$

increases by one or more time steps for every single group having this particular configuration. Thus, $T \geq aN$ as $N \to \infty$, where a is a constant given by the probability of the configuration. Since $T \leq N$ is always true, the leading order term of $\langle T \rangle$ must be linear in N.

We note that division into smaller groups always increases boarding time if $c > 1$. In this case, reducing the range of disorder always results in stronger interference between passengers to be seated near each other, as reflected in the strong sensitivity of the scaling exponent to localized disorder. Thus, lack of control is better than row-wise division of groups for $c > 1$ planes (provided that we do not divide passengers column-wise, which was covered by the $c = 1$ case), which is a lesson shared by most of previous studies [5–14].

Vacancy effect — At last, we consider the effect of vacant seats on asymptotic scaling behaviors of $\langle T \rangle$ for a $c = 2$ case (see Fig. 4), where each seat can be vacant with probability ϕ. The same effect was also studied by [7], but only at a fixed plane size.

Figure 4(a) shows the effect of ϕ when there are no arrival time fluctuations. $\langle T \rangle$ scales as N for $\phi < 1/2$ and as $\ln N$ for $\phi > 1/2$, implying that ϕ can interpolate between the scalings for localized disorder observed at $c = 1$ and $c > 1$. The same observation can be made even in the presence of globalized disorder, as shown in Fig. 4(b), where the scaling changes from $\langle T \rangle \sim N$ to $\langle T \rangle \sim N^{1/2}$. The transition point of the scaling exponent is $1/2$ for $c = 2$, which is confirmed to be $\phi_c = 1 - 1/c$ for the general value of c (even including $c = 1$, in which case the scaling is not affected by vacancy). Interestingly, $\langle T \rangle \sim N^{1/2}$ seems to hold exactly at $\phi = \phi_c$, regardless of the nature of sequential disorder.

All the scaling behaviors of boarding time can be summarized as in Table I. In a single-column ($c = 1$) plane, purely localized disorder produces the N^0 scaling, which includes both saturation and logarithmic divergence, and purely globalized disorder leads to the $N^{1/2}$ scaling. The boarding process is more sensitive to globalized disorder, so the $N^{1/2}$ scaling is observed when both kinds of disorder are present. On the other hand, in a multi-column ($c > 1$) plane, purely localized disorder produces the linear scaling, while purely globalized one again yields the $N^{1/2}$ scaling. Since localized disorder is dominant in this case, the linear scaling is observed when both kinds of disorder are present. Finally, increasing the probability of vacant seats ϕ beyond $\phi_c = 1 - 1/c$ changes the scaling behaviors from the multi-column ones to the corresponding single-column ones while keeping the nature of sequential disorder. However, the borderline scaling at ϕ_c seems to be $N^{1/2}$, regardless of the nature of disorder.

To sum up, we have systematically investigated the relationship between the range of disorder in the passenger sequence and the asymptotic scaling behavior of boarding time using a simple boarding model. Our results clarify the origins of different scalings and indicate which type of disorder plays a dominant role. This offers a natural way to incorporate the boarding problem, previously regarded as an engineer's optimization problem, into the domain of physics.

Finally, we add a few remarks on how the results of our study can be further generalized. For example, we can allow fluctuations in the time required for each passenger to sit down. Since the scaling behavior of the longest blocking subsequence does not change, such fluctuations would not affect the scaling behaviors if they are finite and uncorrelated, but diverging or correlated fluctuations may produce interesting changes. Alternatively, we can generalize the range of sequential disorder by applying Gaussian shuffling with $\sigma \sim N^s$ for $0 \leq s \leq 1$, which seems to produce intermediate values of exponents [20]. We also note that the model considered in our study is similar to asymmetric simple exclusion process (ASEP) [21], which naturally raises the question of how the concepts of ASEP can be applied to the case of boarding. We hope these points to be satisfactorily addressed in future studies.

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[1] A.-L. Barabási and H. E. Stanley, *Fractal Concepts of Surface Growth* (Cambridge University Press, Cambridge, 1995).
[2] V. Sood and S. Redner, Phys. Rev. Lett. **94**, 178701 (2005).
[3] S. Hwang, D.-S. Lee, and B. Kahng, Phys. Rev. Lett. **109**, 088701 (2012).
[4] S. Marelli, G. Mattocks, and R. Merry, Boeing Aero Mag. **1** (1998).
[5] H. Van Landeghem and A. Beuselinck, Eur. J. Oper. Res. **142**, 294 (2002).
[6] M. H. L. van den Briel, J. R. Villalobos, G. L. Hogg, T. Lindemann, and A. V. Mulé, Interfaces **35**, 191 (2005).
[7] P. Ferrari and K. Nagel, Transp. Res. Rec.: J. Transp. Res. Board **1915**, 44 (2005).
[8] M. Bazargan, Eur. J. Oper. Res. **183**, 394 (2007).
[9] J. H. Steffen, J. Air Transp. Manage. **14**, 146 (2008).
[10] D. C. Nyquist and K. L. McFadden, J. Air Transp. Manage. **14**, 197 (2008).
[11] E. Bachmat, D. Berend, L. Sapir, S. Skiena, and N. Stolyarov, J. Phys. A: Math. Gen. **39**, L453 (2006).
[12] E. Bachmat, D. Berend, L. Sapir, and S. Skiena, Adv. Appl. Probab. **39**, 1098 (2007).
[13] E. Bachmat and M. Elkin, Oper. Res. Lett. **36**, 597 (2008).
[14] E. Bachmat, D. Berend, L. Sapir, S. Skiena, and N. Stolyarov, Oper. Res. **57**, 499 (2009).
[15] V. Frette and P. C. Hemmer, Phys. Rev. E **85**, 011130 (2012).
[16] N. Bernstein, Phys. Rev. E **86**, 023101 (2012).
[17] A. M. Vershik and C. V. Kerov, Dokl. Acad. Nauk, **233**, 1024 (1977).
[18] J.-D. Deushel and O. Zeitouni, Ann. Prob. **23**, 852 (1995).
[19] We can still write T instead of $\langle T \rangle$ in the scaling relationships if T is self-averaging. Our results suggest that this may indeed be the case [20].
[20] See Supplemental Material at [URL will be inserted by publisher] for some extra results about effective scaling exponents, self-averaging properties, and effects of the generalized range of sequential disorder.
[21] J. T. MacDonald, J. H. Gibbs, and A. C. Pipkin, Biopolymers **6**, 1 (1968); B. Derrida, Phys. Rep. **301**, 65 (1998); J. Krug, Braz. J. Physics, **30**, 97 (2000); R. A. Blythe and M. R. Evans, J. Phys. A Math. Theor. **40**, R333 (2007).